About some problems arising from large scale parallelization of NP class combinatorial problems

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Example class: the Maximum Clique Problem

Let G be a finite, simple graph: $G = (V, E)$, and C be a subgraph, which has the nodes $\Delta \subseteq V$, and C is spanned by Δ.

- We call the subgraph C a clique, if all of its nodes are connected to each other: $\forall v_i, \forall v_j \in \Delta, i \neq j : (v_i, v_j) \in E$
- We call the size of the clique the number of the nodes in the clique
- we call the clique size of the graph the size of its biggest clique (maximum clique), and denote it by $\omega(G)$.

We can search for the size of a maximum clique, or ask if a given graph has a clique of size k:

- The k-clique decision problem is a well known NP-complete problem
- The maximum clique optimization problem is NP-hard

Is there any real difference between the two problem?

Each algorithm solves the other problem as well:

- finding a maximum clique of size ω will answer if there is a clique of size k (is k smaller or equal, or is it bigger?)
- Using k-clique search and an upper bound (coloring) one can construct a trivial maximum clique search algorithm:

Require: k =an upper bound

```
1:  function kCLIQUE-SEQ(G = (V, E))
2:      FOUND ← false
3:      while ¬FOUND do
4:          FOUND ← kCLIQUE(V, k)
5:          if ¬FOUND then
6:              k ← k − 1
7:          end if
8:      end while
9:      return k
10: end function
```

Yes, we think there are huge differences!

For exact solution of the maximum clique problem a backtracking algorithm used.

For example the Carraghan–Pardalos algorithm is a classical Branch-and-Bound technique. We take the nodes of the graph each by one, and reduce the graph to their neighborhood.
If the reduced graph is *"not satisfactory"* we go back, if it is *"satisfactory"* we do the same (go forward).

- branching: we try several different nodes, if they should be in a maximum clique
- bound: we try to prune the branches of the search tree (number of nodes, coloring, Lovász' θ)

Carraghan–Pardalos Algorithm (1990)

C is the nodes of the clique we are building, C^* are the nodes of the biggest clique found till now. P are the prospective nodes.

Require: $C = \emptyset, C^* = \emptyset, P = V$

```
1:  function CP(C, P)
2:      if |C| > |C*| then
3:          C* ← C
4:      end if
5:      if |C| + |P| > |C*| then
6:          for all vertex p ∈ P do
7:              CP(C ∪ {p}, P ∩ N(p))
8:              P ← P \ {p}
9:          end for
10:     end if
11: end function
```

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Problems

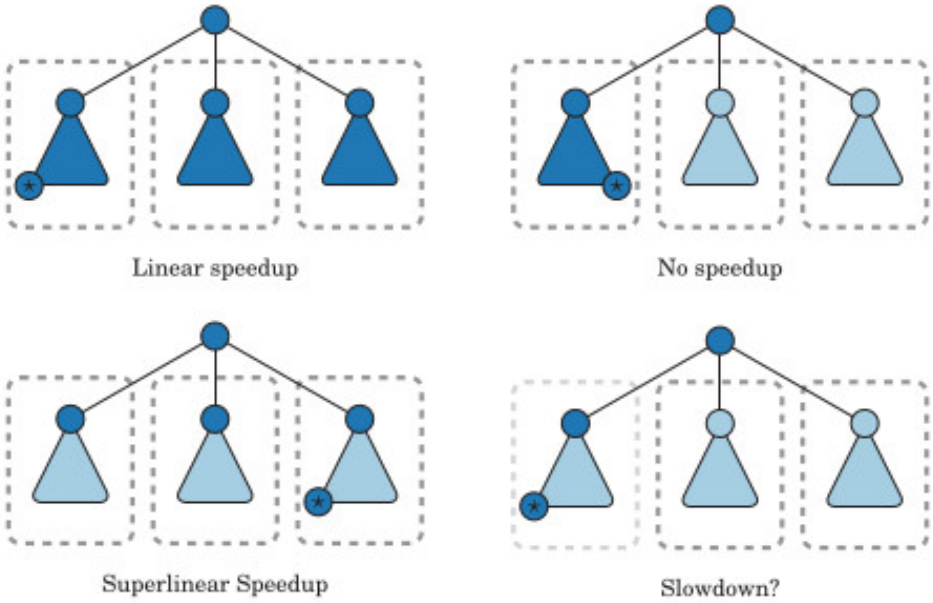
The maximum clique problem has several disadvantages:

1. We branch on the whole P.
2. Finding early a good (big) C^* is crucial.
   (Why do algorithms not searching it heuristically in the beginning?)
3. Heuristics (usually node ordering) for finding a big clique and proving the nonexistence of a bigger one differ.
4. Basically there are two goals at once contradicting each other.
   (Finding a solution and proving the nonexistence.)

Parallel – uneven distribution, superlinear speedup

Using parallel branching leads to extremely uneven distribution.
$\rightarrow$ Difference in several magnitudes.
Branching on first level is good, on second level acceptable, on third level usually does not work. Is other parallelization possible?

Sometimes we see a superlinear speedup. Is it good?



"Superlinear speedup of efficient sequential algorithm is not possible"
1986. Faber, Lubeck, White.

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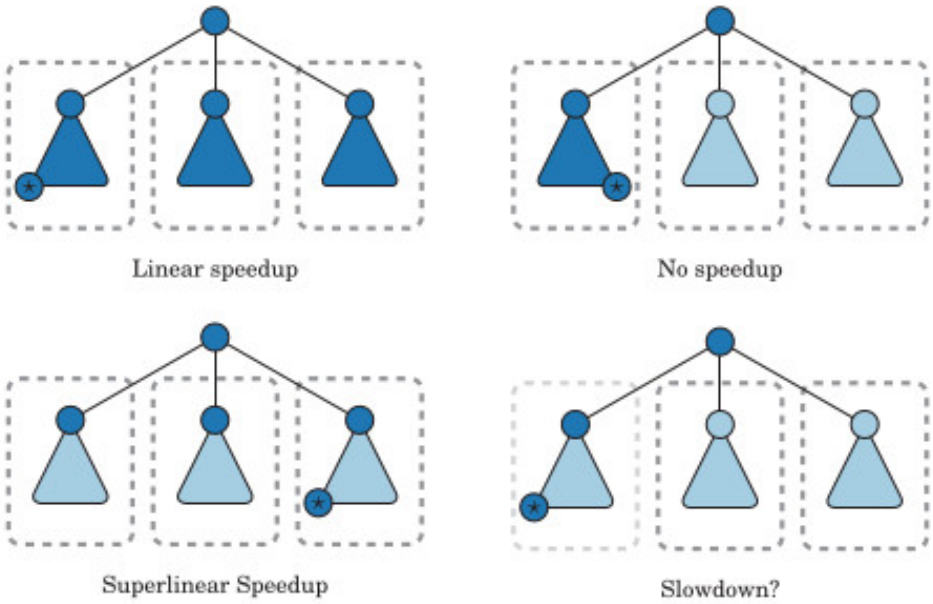

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$k$clique – advantages

```
1:  function kCLIQUE(P, k)
2:      if k = 0 then return true
3:      end if
4:      KCCNS ← construct a k-clique covering node set
5:      for all vertex p ∈ KCCNS do
6:          if kCLIQUE(P ∩ N(p), k − 1) then return true
7:          end if
8:          P ← P \ {p}
9:      end for
10:     return false
11: end function
```

1. Only nonexistence is the goal
2. We can do a better branching (smaller branching factor – Knuth)
3. There is a good estimate for the size of the search tree (Knuth)
4. Can use good ordering of nodes $\rightarrow$ reduce search tree (SAT)

Compared to the 3 best state-of-the-art programs

name	$\|V\|$	%	$\omega(G)$	*k*clique -seq	*k*clique, $k = \omega(G)+1$	BBMC -X	M-clq -13	mcqd -dyn
brock800_3	800	65	25	8837	1955	2452	5561	4290
brock800_4	800	65	26	7277	1543	1787	7037	3072
latin_square_10	900	76	90	213	131	*	*	1180
keller5	776	75	27	53	46	*	238	18098
MANN_a45	1035	99	345	*	*	82	123	2058
sanr200_0.9	200	90	42	141	55	21	2	30
sanr400_0.7	400	70	21	419	140	105	117	110
monoton-7	343	79	19	12	3	31	6	72
monoton-8	512	82	23	846	576	15049	1279	19272
1dc.256	256	88	30	8	2	2	5	22
2dc.1024	1024	68	16	178	112	40	199	146
frb30-15-1	450	82	30	0	0	1613	0	2541
frb35-17-1	595	84	35	2	0	*	1	*
frb40-19-1	760	86	40	17	0	*	1	*
frb45-21-1	945	87	45	839	0	*	119	*
frb50-23-1	1150	88	50	1351	0	*	764	*
frb53-24-1	1272	88	53	42161	0	*	4771	*
frb59-26-1	1534	89	59	*	0	*	*	*
evil-myc11x14	154	94	28	14396	14256	66	235	11563
evil-myc5x36	180	97	72	590	396	2	0	6
evil-myc23x8	184	90	16	645	624	88	23434	1390
evil-s3m25x8	200	92	32	*	*	38987	1206	18148

Table: Running time results in seconds. The "*" sign indicates that the running times are exceeding the 12 hour limit.

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A k-clique covering node set can be used for parallelization as well!

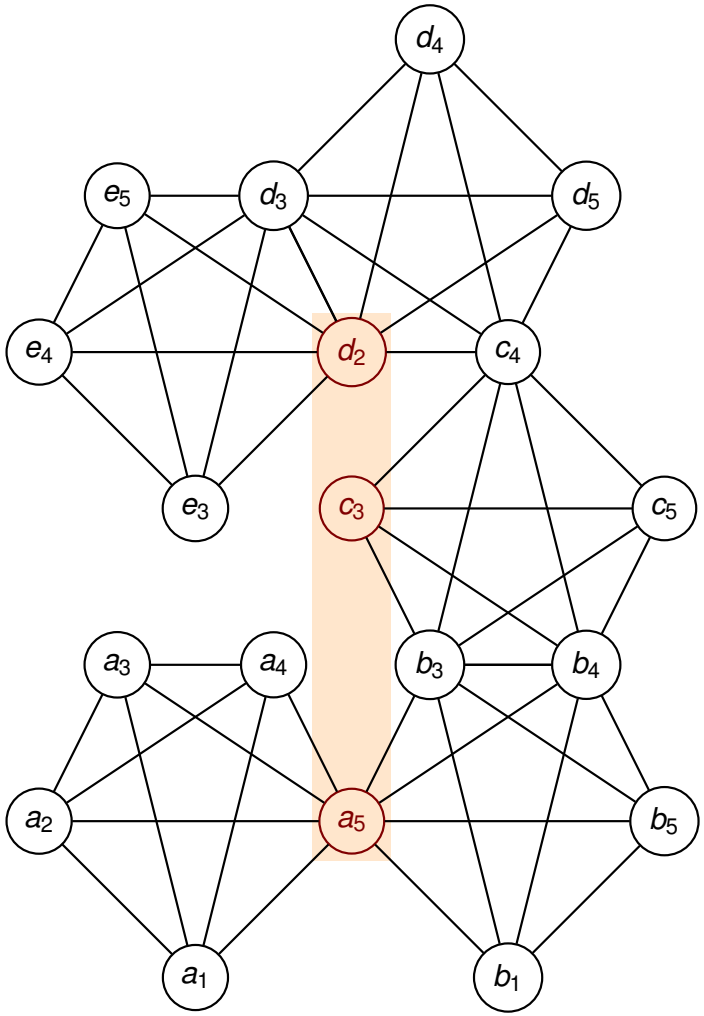

Figure: 5-clique covering node set.

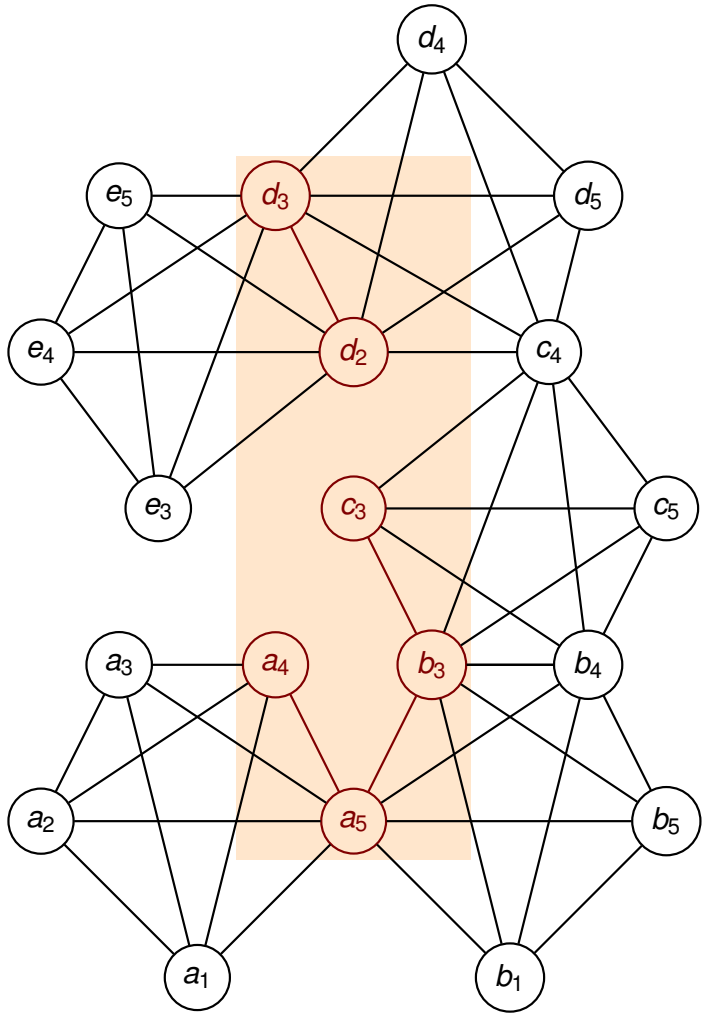

Figure: 5-clique covering edge set.

Estimation of tree size

Knuth, 1975.
Monte Carlo method: random walks in the tree.
Only in k-clique search!
(The tree does not change, as bounds do not change.)

- to know in advantage if the problem is solvable;
- to know the advancement of the program;
- to know which are the hard problems
  - for splitting up
  - for applying more preconditioning.

Las Vegas ordering of sub-problems

The solution of one sub-problem – *there is no* $(k-2)$ *clique in the subgraph of* $N(a,b)$ – effects other sub-problems $\rightarrow$ the edge $\{a,b\}$ can be deleted from all other problems!

The exact sequence of the sub-problems alters the size of the search space. Heuristically it is better to solve the easier problems first.

Question: which problems are easy?

The Las Vegas approach: start all sub-problems parallelly, and those who finish early will "help" others (deletion of a node or edge).
$\rightarrow$ "Work-Help parallelization"

(Other method: see tree size estimation!)

Las Vegas ordering – example

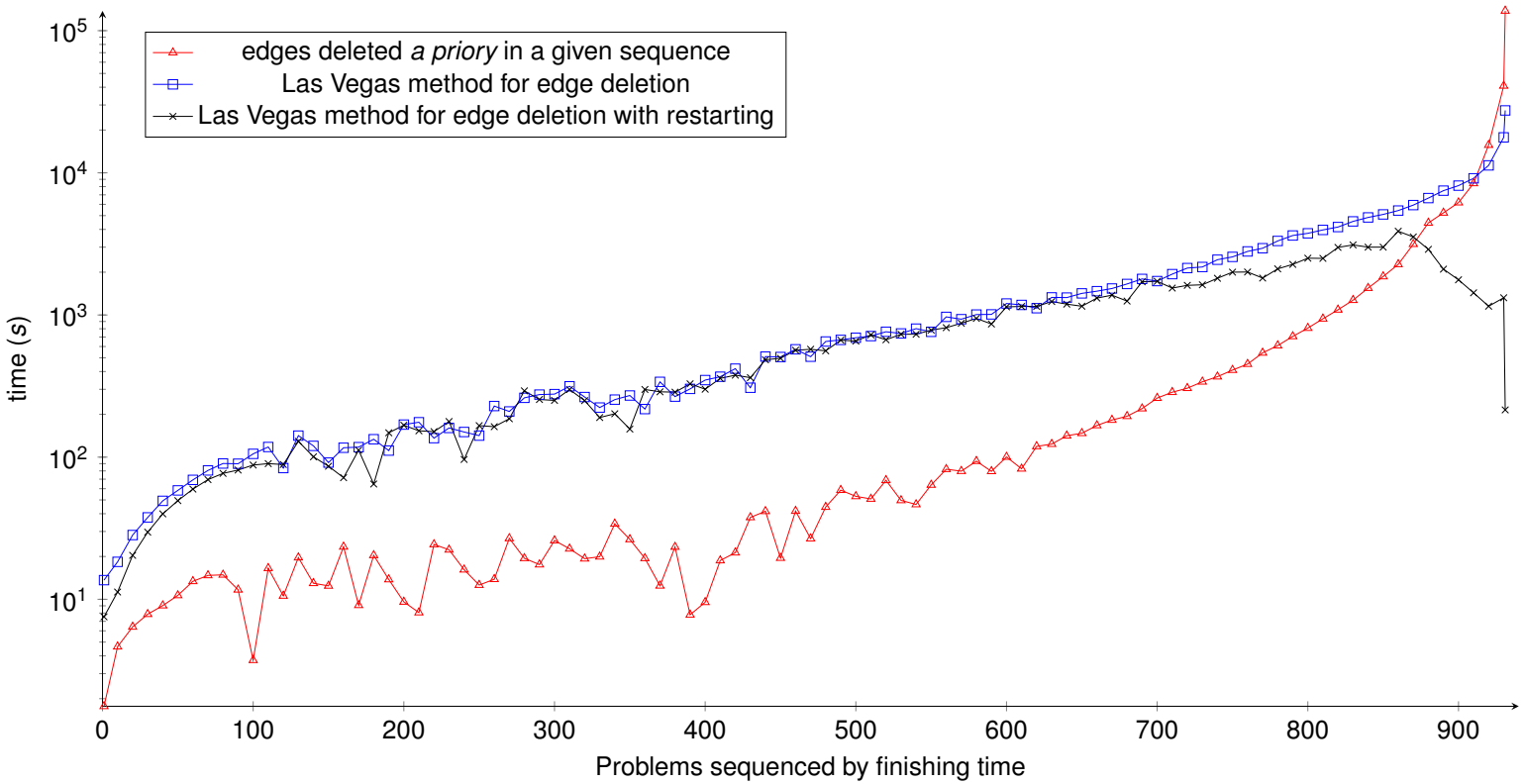

Figure: The time sequence of running times of the `monoton-9` subproblems.

Las Vegas ordering – example

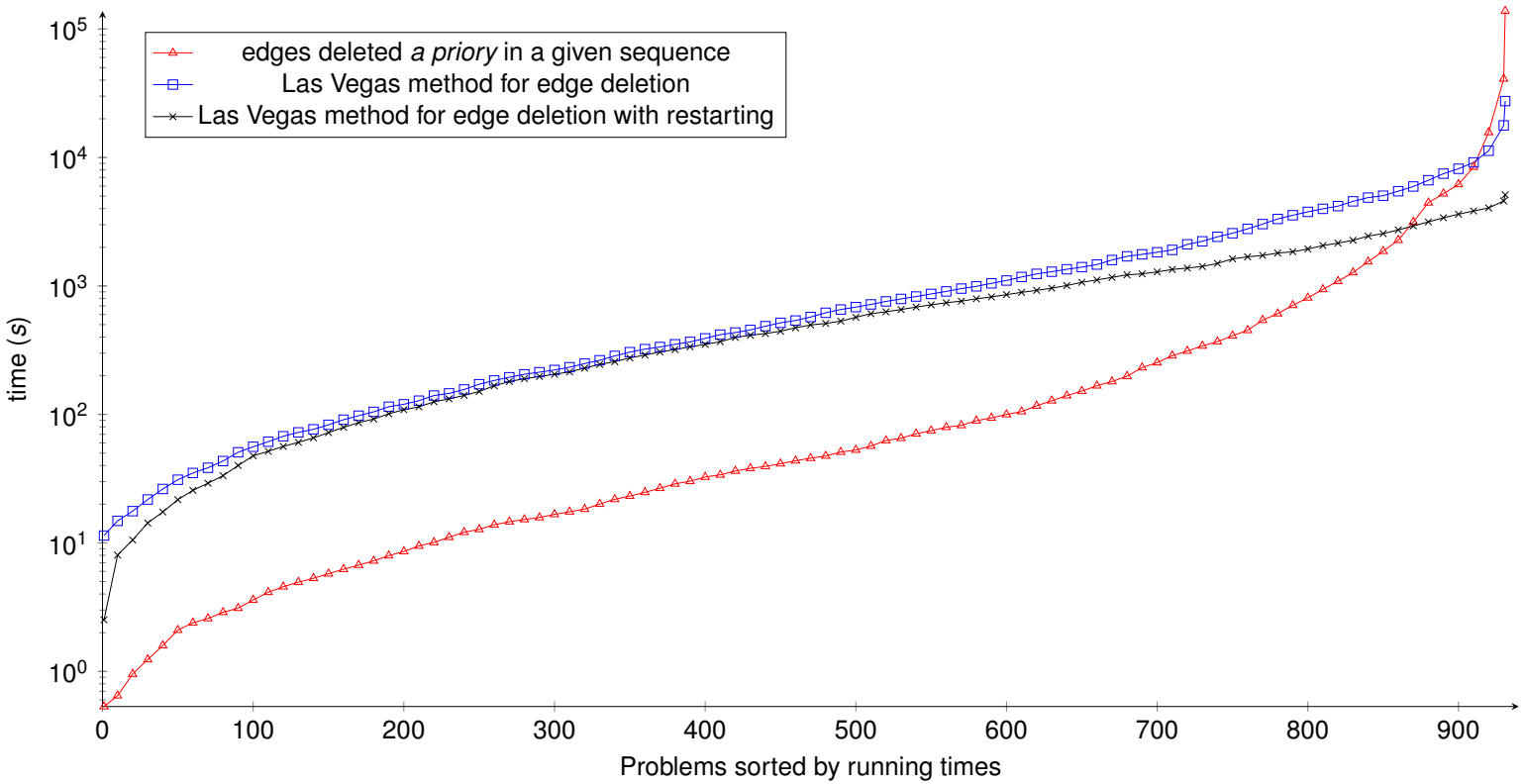

Figure: The sorted running times of the `monoton-9` subproblems.

Conclusion

- Proved to be extremely efficient against state-of-the art programs that use a much better upper bound;
- May be used for solving other combinatorial optimization problems – coloring, scheduling, subgraph isomorphy;
- Some problems better modelled for k-clique;
- Opens new possibilities for subproblem generation $\rightarrow$ new ways of parallelization;
- Was used efficiently for parallelizing on up to 500 cores;
- No superlinear speedup!
- Additional techniques allowed for parallelization on up to 70k cores,

Thank you for your attention!

Questions?

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